Merger Remedies in Multimarket Oligopoly*

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Abstract

A merger will often affect several different product and geographic markets. In many cases the merger is cleared subject to remedies being implemented in some of those markets. This paper develops a framework in which an antitrust authority bargains with merging firms over which remedies to implement. Remedies are modeled as asset divestitures. The antitrust authority maximizes aggregate consumer surplus, subject to the constraint that consumer surplus in each market must exceed some threshold. We first solve for the "minimum divestitures" needed to meet the market-level constraints, and provide conditions under which they are lower in more competitive markets. We then show that the solution to the cross-market bargaining problem is bang-bang. In particular, in a given market it is optimal to implement either the minimum divestitures, or else divest as many assets as possible. We also provide conditions under which these "maximal divestitures" are implemented in more competitive markets.

Keywords: Antitrust, merger policy, structural remedies, divestitures, oligopoly.

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1 Introduction

Antitrust authorities regularly clear mergers subject to the implementation of remedies. For example in the U.S., more than 60% of the mergers that were challenged by the U.S. authorities between 2003 and 2012 were later approved subject to remedies (Kwoka, 2014). Similarly, between 1990 and 2014, the European Commission was around 15 times less likely to prohibit a merger, than it was to clear it subject to remedies (Affeldt, Duso, and Szücs, 2018). Although it varies by jurisdiction, there is often some form of negotiation between the authorities and the merging parties as to exactly which remedies should be implemented.[1] At the same time, antitrust authorities typically have a preference for 'structural' remedies whereby the merging parties divest assets such as manufacturing facilities and personnel to their competitors, or remedies that involve the licensing of intellectual property.[2]

Importantly, a merger will often affect several different markets. For example, consumers in different geographic markets are affected by a merger between two supermarket chains, while consumers in different drug product markets are affected by a merger between two pharmaceutical companies. However, in many cases, a merger only raises concerns in a subset of those markets. In principle, antitrust authorities typically use a so-called 'market-by-market' approach. This means that they only approve a merger if, after imposing remedies, competition is restored in each individual market. In practice, however, authorities may deviate from this approach. Firstly, some jurisdictions such as the UK and Germany explicitly leave open the possibility of 'balancing' gains and losses across markets.[3] This may be appropriate when, for example, the same consumers are present in many of the affected markets (Crane, 2015).[4] Secondly, Farrell (2003) argues that authorities may 'over-fix', by using remedies to either solve pre-existing competitive problems in an industry, or in individual markets which are not directly affected by the merger.[5] Finally, even if the authorities do use a market-by-market approach, the merging parties might wish to speed up approval of the merger (and

[1]In Canada, for example, "a remedy is consensually negotiated between the agency and the parties to the transaction" (OECD, 2016). Meanwhile in the U.S., "the [merging] parties and [FTC] staff negotiate a proposed settlement and finalize terms" (FTC, 2012), although whether the remedy is ultimately approved depends on a vote amongst the Commissioners.

[2]Another type of remedy is 'behavioral' comitments by the merging parties about future market conduct. Structural remedies are usually preferred since they are easier to implement and do not require monitoring (see CMA (2018) for example).

[3]In the UK, the CMA can clear a merger at Phase 1 if a significant lessening of competition (SLC) and its adverse effects in one market are outweighed by consumer gains "in any market in the United Kingdom (whether or not in the market(s) in which the SLC has occurred or may occur)." (CMA, 2018) See also OECD (2016) for discussion of the so-called "balancing clause" in German merger law. Werden (2017) points out that even the U.S. merger guidelines allow the market-by-market rule to be broken in certain circumstances.

[4]Werden (2017) also argues that a market-by-market approach is problematic "if a merger with massive competitive benefits would be made unlawful by unfixable anticompetitive effects in a single tiny market."

[5]A possible example is the recent merger between American Airlines and US Airways, where gates and slots at key airports were divested to low-cost carriers. The DOJ argued that this would "disrupt the cozy relationships among the incumbent legacy carriers", potentially "shift the landscape of the airline industry" and "enhance system-wide competition". See https://www.justice.gov/opa/pr/justice-department-requires-us-airways-and-american-airlines-divest-facilities-seven-key

thus benefit more quickly from resulting cost savings) by offering more remedies than are strictly necessary (Ormosi, 2009).

In this paper, we offer a tractable framework to think about merger remedies. Our baseline model has a continuum of independent and heterogeneous markets. In each market where it is active, a firm is endowed with a market-specific vector of assets which then determines its marginal cost in that market. Firms compete in each market à la Cournot. A merger between two or more of the firms may generate market-specific synergies, as well as a set of market-specific assets which could be divested to competitors. We place relatively weak restrictions on how (divested) assets affect firm costs. The merger (plus any asset divestitures) must be cleared by an antitrust authority which seeks to maximize aggregate consumer surplus, subject to two constraints. The first constraint is that post-merger consumer surplus in each market must exceed some fraction x of the pre-merger consumer surplus. A natural benchmark is $x = 1$, whereby no market is made worse off by the merger. However, our analysis also allows for both $x < 1$ and $x > 1$. The fomer is consistent with a merger not significantly reducing competition in any market[6], while the latter is consistent with the 'over-fixing' that we described in the previous paragraph. We also allow x to differ across markets.[7] The second constraint on the antitrust authority is that it must negotiate divestitures with the merging parties. We model this via efficient bargaining, and show that it nests both a market-by-market approach, as well as the case where the authority can balance pro- and anti-competitive effects of the merger across markets.

We first solve for the "minimum divestitures" that are required to meet the market-level consumer surplus constraints. In doing this, we introduce two concepts. The first is a *synergies curve*, which varies the level of synergies in a given market, and traces out the resulting combinations of consumer surplus and merged firm profit. The second concept that we introduce is a *divestitures curve*, which fixes the level of synergies, but varies both which assets are divested and to whom, and again traces out the resulting combinations of consumer surplus and (maximized) merged firm profit. Using these two concepts, we then solve for the best divestitures (from the merged firm's point of view) that meet the market-level constraints. We also show that provided x is not too far above one, markets that are more competitive pre-merger require fewer (if any) divestitures.

We then consider the multimarket bargaining problem between the antitrust authority and the merging parties. In the special case where the merging parties have all the bargaining power, only minimal divestitures are implemented. In other cases larger divestitures are (typically) implemented, and the optimal solution 'balances' their effect across markets. In order to characterize this solution, we introduce the concept of a *remedies exchange rate*. Intuitively, this measures how many dollars the merged firm must give up in a market when consumer surplus is increased by one dollar through divestitures. We show that under certain conditions on the curvature of demand, this exchange rate improves as more and more assets

[6]In the U.S., the Clayton Act prohibits only mergers that lead to a *substantial* lessening of competition, while in the European Union, only mergers that *significantly* impede effective competition are outlawed.

[7]The antitrust authority might, for example, use a higher x in markets with poor or vulnerable consumers.

are divested. As a result, the optimal solution is bang-bang: in some markets only the minimal divestitures are implemented, whereas in others as many assets as possible are divested. We derive conditions under which these "maximal divestitures" are larger in more competitive markets, and also examine how a change in the authority's bargaining power affects the set of markets with maximal divestitures.

Related Literature Much of the literature on mergers focuses on a single market, and assumes that the authority must either accept or reject the merger. At the heart of many of these papers is the Williamson (1968) trade-off, whereby a merger can raise prices due to a market power effect, or lower them due to efficiencies. Farrell and Shapiro (1990) formalize this trade-off in a homogeneous goods Cournot model, and show that without synergies the market power effect dominates, leading to lower consumer surplus. More recently, Nocke and Whinston (2010) study merger policy in a single-product Cournot setting, where mergers can be proposed sequentially over time, and provide conditions under which a myopic rule is dynamically optimal. Meanwhile Nocke and Whinston (2013) show that an antitrust authority optimally discriminates against larger mergers when firms can choose which merger to propose. Different from these papers, Johnson and Rhodes (2019) and Nocke and Schutz (2019) consider mergers in a multiproduct setting. The former studies mergers between vertically differentiated firms in a Cournot setting, and finds for example that mergers without synergies can raise consumer surplus, but only when certain necessary observable conditions on the pre-merger industry structure are satisfied. The latter studies mergers between multiproduct firms in a differentiated Bertrand setting, and among other things provides theoretical support for using the naively-computed change in the Herfindahl index as a screen in merger review. However none of these papers consider merger remedies.

Some recent papers do investigate structural merger remedies. Vergé (2010) considers a homogeneous goods Cournot model and provides conditions under which remedies cannot undo the competitive harm of a merger in the absence of synergies. Vasconcelos (2010) studies the extent to which 'over-fixing' may arise in the equilibrium of a dynamic merger review game. Meanwhile Cosnita-Langlais and Tropeano (2012) examine whether merger partners that are privately-informed about the magnitude of any synergies, can credibly signal that information through the remedies that they propose to the antitrust authority.

The rest of the paper proceeds as follows. Section 2 describes the model, while Section 3 recaps some standard analysis of Cournot models. Section 4 then analyzes outcomes at the level of an individual market, while Section 5 studies the cross-market bargaining problem. Finally, Section 6 concludes with a discussion of future avenues for research. All omitted proofs are available in the appendix.

2 Model

There is a continuum of independent markets, indexed by $j \in [0,1]$, and a set $\mathcal{N} = \{1, ..., N\}$ of firms, each of which is present in a subset of markets. The set of firms present in market j is denoted $\mathcal{N}^j \subset \mathcal{N}$, with $n^j \equiv |\mathcal{N}^j|$. Within each market, firms produce a homogeneous good with constant returns to scale and compete in a Cournot fashion. The marginal cost of firm i in market j is denoted c_i^j. In market j, firms face inverse market demand $P^j(Q)$, where Q denotes market-level output. We impose standard assumptions on demand ensuring that there exists a unique Nash equilibrium in quantities for any vector of marginal costs: $P^j(0) > 0$, $\lim_{Q\to\infty} P^j(Q) = 0$, and for all Q such that $P^j(Q) > 0$, $P^{j\prime}(Q) < 0$ and $P^{j\prime}(Q) + QP^{j\prime\prime}(Q) < 0$. The last inequality is equivalent to requiring that $\sigma^j(Q) < 1$, where $\sigma^j(Q) \equiv -QP^{j\prime\prime}(Q)/P^{j\prime}(Q)$ is the curvature of inverse demand.

We consider a merger among the firms in set $\mathcal{M} \subset \mathcal{N}$, where $|\mathcal{M}| \geq 2$. The set of non-merging outsiders is denoted $\mathcal{O} \equiv \mathcal{N} \setminus \mathcal{M}$. Following the merger, there are $\overline{n}^j \equiv |\mathcal{O}^j|+1$ firms present in market j. For the merger to be approved, the antitrust authority may require the merger partners to divest some of their assets. The t-dimensional vector of assets that could feasibly be divested in market j is denoted $K^j \in \mathbb{R}^t_+$. For simplicity, we treat the amount of each asset as a continuous variable. The vector of assets divested to outsider $i \in \mathcal{O}$ in market j is denoted Δ_i^j, where $\sum_{i\in\mathcal{O}} \Delta_i^j \leq K^j$. We denote outsider i's post-merger marginal cost by $\overline{c}_i^j(\Delta_i^j)$, where $\overline{c}_i^j(0) = c_i^j$. Similarly, we denote the merged firm's marginal cost in market j by $\overline{c}_M^j\left(-\sum_{i\in\mathcal{O}} \Delta_i^j\right)$. In the special case where $\overline{c}_M^j(0) = \min_{i\in\mathcal{M}} c_i^j$, absent divestitures, the merger does not involve any synergies in the sense of Farrell and Shapiro (1990). Throughout we assume that post-merger marginal costs, $\overline{c}_M^j(-\sum_{i\in\mathcal{O}} \Delta_i^j)$ and $\overline{c}_i^j(\Delta_i^j)$ for each $i \in \mathcal{O}$, are weakly decreasing and $\mathcal{C}^2$. For now, we assume that the merger partners do not receive any revenue from divesting assets.

Let Q^{j*} and $\overline{Q}^{j*}((\Delta_i^j)_{i\in\mathcal{O}})$ denote the equilibrium output in market j before and after the merger, respectively. Consumer surplus in market j as a function of market-level output Q is given by

$$v^j(Q) = \int_0^Q [P^j(z) - P^j(Q)]dz.$$

Equilibrium consumer surplus, aggregated over all markets, is denoted by $V^* \equiv \int_{[0,1]} v^j(Q^{j*})dj$ before the merger and by $\overline{V}^* \equiv \int_{[0,1]} v^j(\overline{Q}^{j*})dj$ after the merger. The market-j equilibrium profit of a firm with marginal cost c when market-level equilibrium output is Q is denoted by $\pi^j(Q; c)$. The pre-merger equilibrium profit of firm i, aggregated over all markets, is denoted by $\Pi_i^* \equiv \int_{[0,1]} \pi^j(Q^{j*}; c_i^j)dj$. Similarly, the aggregate equilibrium profit of the merged firm is denoted by $\overline{\Pi}_M^* \equiv \int_{[0,1]} \pi^j(\overline{Q}^{j*}; \overline{c}_M^j)dj$.

We assume that the antitrust authority applies a consumer surplus standard. The set of divestitures, $(\Delta_i^j)_{i\in\mathcal{O}}^{j\in[0,1]}$, is determined through an efficient bargaining process between the authority and the merger partners, subject to two constraints. First, the merger must be privately profitable, i.e. $\overline{\Pi}_M^* \geq \sum_{i\in\mathcal{M}} \Pi_i^*$. Second, the post-merger consumer surplus in each market j must exceed a fraction x^j of the pre-merger level, i.e. $v^j(\overline{Q}^{j*}) \geq x^j v^j(Q^{j*})$.[8] In the special case $x^j = 1$ consumers cannot be made worse off in any market. However, as discussed in the introduction, x^j could also differ from one and vary across markets.

3 Preliminary Analysis

This section briefly recaps standard analysis of Cournot oligopoly in a given market $j \in [0,1]$. To ease notation we drop both the firm subscript i and the market subscript j.

A firm with marginal cost c chooses output q to maximize its profit $(P(Q) - c)\,q$. The firm's *output fitting-in function* $r(Q;c)$ is the output level that solves its first-order condition:

$$r(Q;c) = \begin{cases} \frac{P(Q)-c}{-P'(Q)} & \text{if } c < P(Q), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We can then use the output fitting-in function to rewrite the expression for firm profit, and thereby obtain the firm's *profit fitting-in function*:

$$\pi(Q;c) = -\left(r(Q;c)\right)^2 P'(Q). \tag{2}$$

Notice that given our assumptions on market demand $P(Q)$, both $r(Q;c)$ and $\pi(Q;c)$ are weakly decreasing in Q and c, and strictly so whenever $c < P(Q)$.

The pre-merger market-level equilibrium output Q^* is then the unique Q which solves

$$Q = \sum_{i \in \mathcal{N}} r(Q; c_i), \tag{3}$$

while the post-merger market-level equilibrium output $\overline{Q}^*$ is the unique solution to

$$Q = r\left(Q; \overline{c}_M\left(-\sum_{i \in \mathcal{O}} \Delta_i\right)\right) + \sum_{i \in \mathcal{O}} r(Q; \overline{c}_i(\Delta_i)). \tag{4}$$

4 Market-Level Analysis

This section introduces a framework to analyze the effect of merger synergies and divestitures at the level of an individual market. Using this framework, we solve for the *minimum divestitures* that the merger partners must offer in order to meet the authority's market-level consumer surplus constraint. We then compare these divestitures across markets.

[8] The analysis would remain qualitatively unchanged if the antitrust authority also faced an "aggregate constraint", e.g. requiring that consumer surplus, aggregated over all markets, cannot decrease.

4.1 Minimum Divestitures

We begin by analyzing mergers in the absence of divestitures. To ease notation, we drop the market subscript j whenever no confusion arises. The following (standard) result shows how market outcomes vary with the merged firm's marginal cost $\overline{c}_M(0)$:

Lemma 1. *A small increase in the post-merger marginal cost $\overline{c}_M(0)$ strictly decreases both equilibrium consumer surplus $v(\overline{Q}^*)$ and the merged firm's equilibrium profit $\pi(\overline{Q}^*;\overline{c}_M(0))$ if $\overline{c}_M(0) < P(\overline{Q}^*)$, and has no effect on either otherwise.*

Intuitively, if the merged firm produces strictly positive output, then an increase in its marginal cost—holding fixed the output of outsiders—reduces the firm's profit and induces it to produce less. Outsider firms respond by increasing their output levels, further hurting the merged firm. Moreover, given the properties of $P(Q)$, the latter output effect is dominated by the former. Consequently market-level output falls, and therefore so does consumer surplus.

Lemma 1 implies that in the absence of divestitures, by varying the merged firm's synergy ($\min_{i\in\mathcal{M}} c_i - \overline{c}_M(0)$) we trace out an upward-sloping curve in $(v(\overline{Q}^*), \pi(\overline{Q}^*;\overline{c}_M))$ space. We call this the *synergies curve*, and illustrate it in Figure 1. In this figure, the diamond depicts the pre-merger outcome, while the dashed vertical line depicts the authority's market-level consumer surplus constraint (for the case $x < 1$). We know from Farrell and Shapiro (1990) that without any synergies consumer surplus must decrease after the merger, as illustrated by the leftmost point on the synergies curve. We also know from Nocke and Whinston (2010) that if there exists a synergy such that pre- and post-merger consumer surplus are identical, then at $v(\overline{Q}^*) = v(Q^*)$ the synergies curve lies above the pre-merger outcome, as also illustrated in the figure.

The following is a straightforward corollary of Lemma 1:

Corollary 1. *There exists a $\hat{c}_M(x)$ such that, absent divestitures, the merger strictly decreases consumer surplus if $\overline{c}_M(0) > \hat{c}_M(x)$, and otherwise weakly increases it.*

Figure 1 depicts two possible merged-induced outcomes without divestitures. The one depicted as a circle corresponds to a post-merger cost that satisfies $\overline{c}_M(0) < \hat{c}_M(x)$. Here, even absent divestitures, the merger meets the constraint. On the other hand, the outcome depicted as a triangle corresponds to a post-merger cost that satisfies $\overline{c}_M(0) > \hat{c}_M(x)$. Here, absent divestitures, the market-level constraint is not met.

We now fix $\overline{c}_M(0)$ and focus on the case where $\overline{c}_M(0) > \hat{c}_M(x)$, such that divestitures are required to meet the market-level constraint. We study how consumer surplus and the merged firm's profit change as we vary both the set of divested assets as well as their allocation amongst outsiders. To this end it is useful to define

$$\overline{v}_{ND} \equiv v(\overline{Q}^*(0)), \quad \text{and} \quad \overline{v}_{\max} \equiv \max_{\boldsymbol{\Delta}\text{s.t.} \sum_{i\in\mathcal{O}} \Delta_i \le K} v(\overline{Q}^*(\boldsymbol{\Delta})), \tag{5}$$

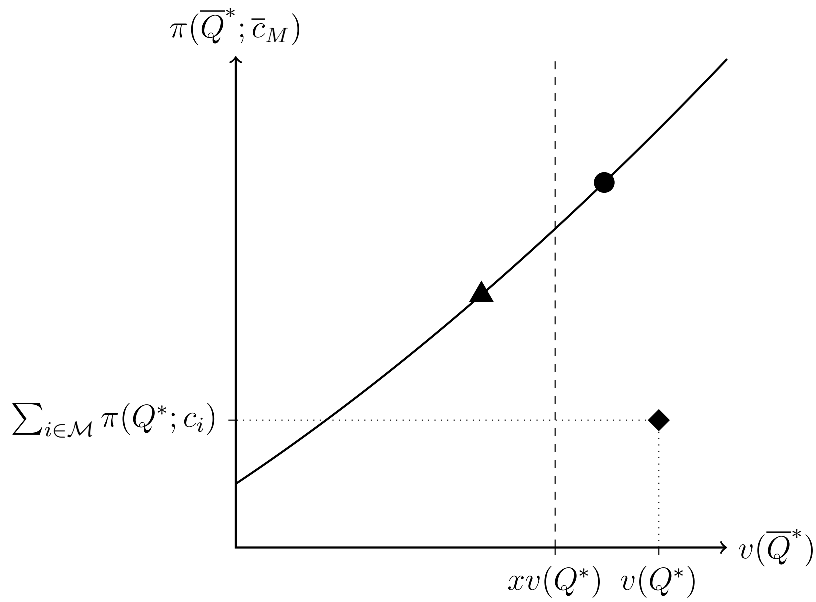

Figure 1: The synergies curve.

where $\mathbf{\Delta} \equiv (\Delta_i)_{i\in\mathcal{O}}$ is a matrix denoting how much of each asset is divested to each outsider. Thus $\overline{v}_{ND}$ is the consumer surplus that arises absent divestitures, and $\overline{v}_{\max}$ is the greatest possible consumer surplus that arises with feasible divestitures. To make the problem interesting we suppose that $\overline{v}_{\max} > \overline{v}_{ND}$. It is instructive to start with a special case, which we call the *licensing / redundant asset case*.

Licensing / redundant assets case We first consider a special case of the model, in which the merged firm's cost $\overline{c}_M(-\sum_{i\in\mathcal{O}} \Delta_i)$ is the same for all feasible divestitures $\sum_{i\in\mathcal{O}} \Delta_i \leq K$. This special case could arise when, for example, the divestible assets consist of intellectual property or are held in duplicate by the merged firm, such that divesting them has no effect on the merged firm's cost (but weakly reduces the cost of an outsider firm that receives them). We obtain the following result:

Lemma 2. *Suppose $\overline{c}_M(\cdot)$ is constant. The set of consumer suplus levels v that can be induced by feasible divestitures is $[\overline{v}_{ND}, \overline{v}_{\max}]$. Moreover there exists a continuously differentiable function $d_M(v)$, such that if a set of divestitures induces consumer surplus v, the merged firm's profit is $d_M(v)$. This function is weakly decreasing, and strictly so for $d_M(v) > 0$.*

We call the function $d_M(v)$ defined in Lemma 2 the *divestitures curve*, and we illustrate it in Figure 2. In the figure we again use a triangle to depict the no-divestitures outcome, which is also the origin of the divestitures curve. The divestitures curve is single-valued because when $\overline{c}_M(.)$ is constant, both consumer surplus and the merged firm's profit only depend on the set of divestitures through the induced equilibrium output $\overline{Q}^*$. The divestitures curve is also downward-sloping because $v(\overline{Q}^*)$ is strictly increasing, whilst $\pi(\overline{Q}^*; \overline{c}_M(0))$ is weakly

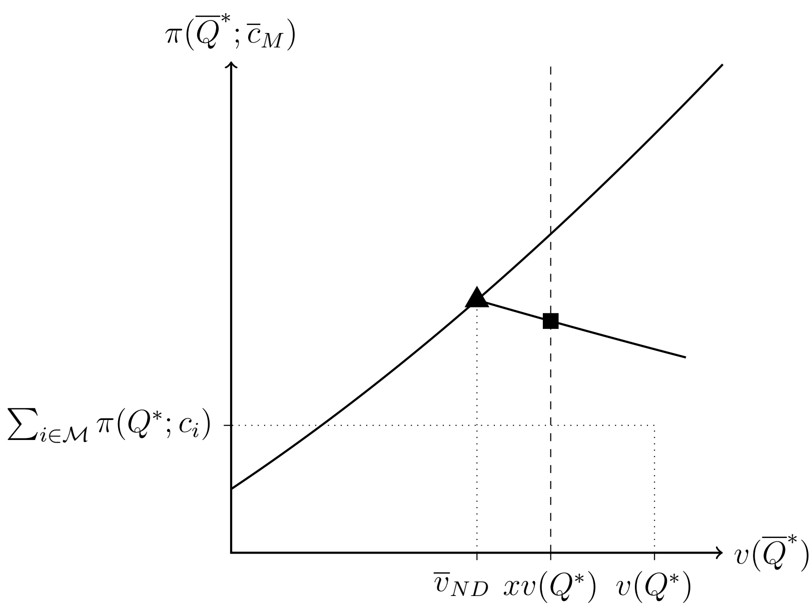

Figure 2: The divestitures curve with licensing/redundant assets.

decreasing in $\overline{Q}^*$ and strictly so provided the merged firm is active. Finally, the support of the divestitures curve is $[\overline{v}_{ND}, \overline{v}_{\max}]$. Since in the licensing/redundant asset case divestitures weakly reduce each firm's cost, consumer surplus must weakly exceed $\overline{v}_{ND}$. Continuity of firms' costs functions then implies that any consumer surplus between $\overline{v}_{ND}$ and the maximum feasible level $\overline{v}_{\max}$ can be induced by some set of divestitures.

General case We now turn to the general case, where the merged firm's marginal cost $\overline{c}_M(-\sum_{i\in\mathcal{O}} \Delta_i)$ is a weakly decreasing function. It is helpful to start with the following thought experiment. Suppose we fix the total assets $\sum_{i\in\mathcal{O}} \Delta_i$ to be divested, and then vary how they are allocated amongst the outsiders. Since the merged firm's cost $\overline{c}_M(-\sum_{i\in\mathcal{O}} \Delta_i)$ does not change as we vary the allocation, this exercise is analogous to the one conducted above for the licensing/redundant asset case. We can therefore define a *conditional divestitures curve* $\tilde{d}_M(v; \sum_{i\in\mathcal{O}} \Delta_i)$, which gives the merged firm's profit when total assets $\sum_{i\in\mathcal{O}} \Delta_i$ are divested in such a way that they induce consumer surplus v. Closely following the arguments underlying Lemma 2, $\tilde{d}_M(v; \sum_{i\in\mathcal{O}} \Delta_i)$ is single-valued, and both continuously differentiable and (weakly) decreasing in v. Any v satisfying the following inequality can be supported:[9]

$$\begin{array}{c} \min_{\boldsymbol{\Delta}} v(\overline{Q}^*(\boldsymbol{\Delta})) \\ \text{s.t. } \sum_{i\in\mathcal{O}} \Delta_i = D \end{array} \leq v \leq \begin{array}{c} \max_{\boldsymbol{\Delta}} v(\overline{Q}^*(\boldsymbol{\Delta})) \\ \text{s.t. } \sum_{i\in\mathcal{O}} \Delta_i = D \end{array} \tag{6}$$

There exists a whole family of conditional divestitures curves, each of which is indexed by the vector of total divested assets $\sum_{i\in\mathcal{O}} \Delta_i$. Moreover, these curves are 'ordered' in the

[9]Note that a conditional divestitures curve may degenerate to a single point, e.g. when there is only one outsider, and thus only one way to allocate a given set of divested assets.

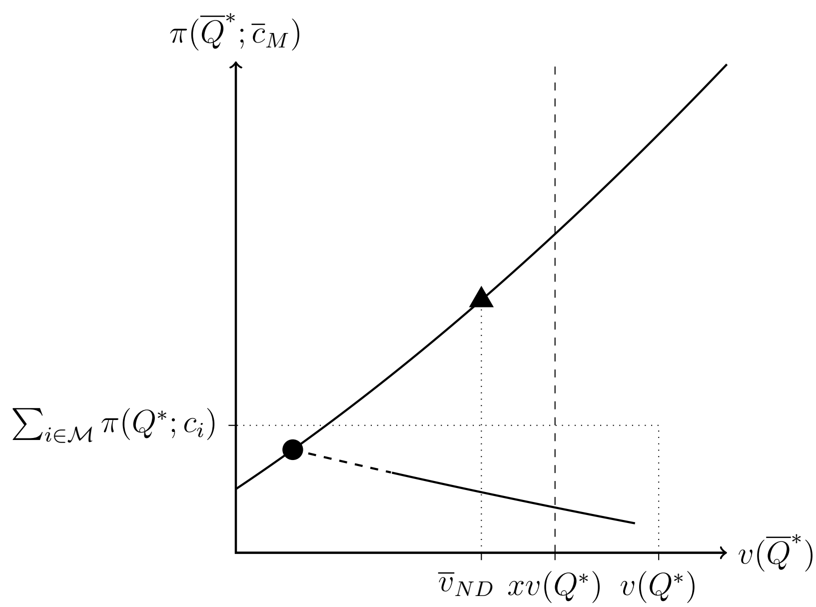

Figure 3: A conditional divestitures curve in the general case.

following sense. Consider two different vectors of total divested assets D' and D'', which induce costs for the merged firm of $\overline{c}_M(-D')$ and $\overline{c}_M(-D'')$ respectively. One possibility is that $\overline{c}_M(-D') = \overline{c}_M(-D'')$, in which case the conditional divestitures curves associated with D' and D'' coincide at each jointly feasible consumer surplus level v. Another possibility, though, is that $\overline{c}_M(-D') > \overline{c}_M(-D'')$, in which case we have that $\tilde{d}_M(v; D') \leq \tilde{d}_M(v; D'')$ at all jointly feasible v, with a strict inequality whenever $\tilde{d}_M(v; D'') > 0$.

Figure 3 illustrates the conditional divestitures curves for one particular vector of total divested assets $\sum_{i\in\mathcal{O}} \Delta_i$. Conceptually, we can think about it using the following two steps. In the first step, imagine that we take the assets $\sum_{i\in\mathcal{O}} \Delta_i$ away from the merged firm, but do not allocate them to any outsider. This is equivalent to starting at the no-divestitures outcome, which is again depicted by the triangle, and then moving down the synergies curve, until reaching the outcome depicted by the circle. In the second step, imagine that we now vary both how many of the $\sum_{i\in\mathcal{O}} \Delta_i$ assets to allocate amongst outsiders, as well as to whom. As we do this, the merged firm's cost is fixed and only post-merger output $\overline{Q}^*$ changes. This is therefore equivalent to starting at the outcome depicted by the circle, and then tracing out a downward-sloping curve in $(v(\overline{Q}^*), \pi(\overline{Q}^*; \overline{c}_M))$-space. The dashed part of the curve corresponds to v that can only be induced when some of the assets are withheld from outsider firms. Meanwhile the solid part of the curve corresponds to v that can be induced when all assets are allocated to outsiders, and is hence the conditional divestitures curve.

Figure 4 then represents, via the shaded area, the whole family of conditional divestitures curves. Notice that several different divestitures may induce the same consumer surplus v, and that the merged firm may not be indifferent between them. Notice also that some

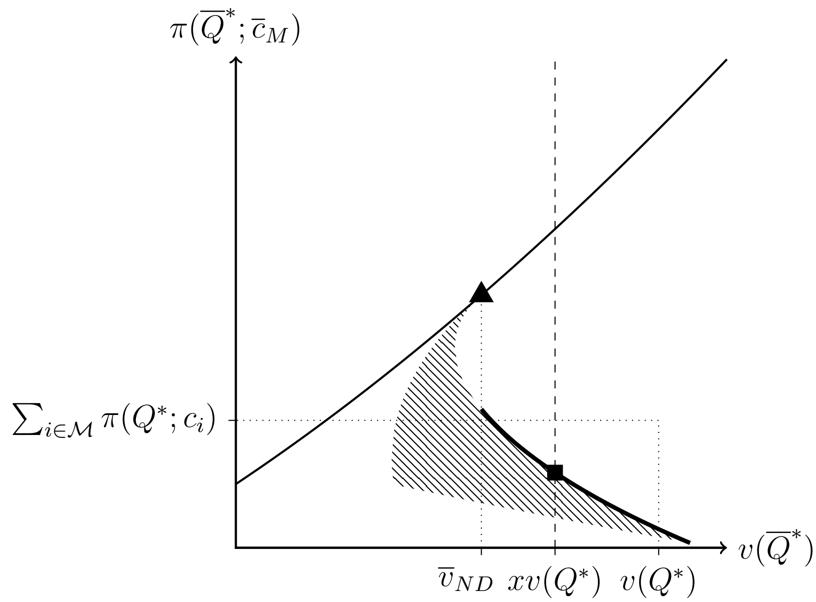

Figure 4: The efficient divestitures curve.

divestitures may reduce consumer surplus below the no-divestitures level $\overline{v}_{ND}$, and thus be dominated for both consumers and the merged firm itself. To this end, we introduce the concept of an *efficient divestitures curve* $d_M(v)$:

Lemma 3. *There exists a function* $d_M(v) \equiv \max_{\boldsymbol{\Delta} s.t. \sum_{i\in\mathcal{O}}\Delta_i\le K} \tilde{d}_M(v;\sum_{i\in\mathcal{O}}\Delta_i)$ *such that for any consumer surplus level* $v\in[\overline{v}_{ND},\overline{v}_{\max}]$*, the maximal profit for the merged firm is* $d_M(v)$*. This function is weakly decreasing, and strictly so for* $d_M(v)>0$*.*

The (efficient) divestitures curve $d_M(v)$ is the upper envelope of the conditional divestitures curves for all v that exceed the no-divestitures level $\overline{v}_{ND}$. It is represented by the solid curve in Figure 4. Notice that in the figure $d_M(v)$ has a downward jump at $v=\overline{v}_{ND}$. Intuitively, such a discontinuity at the no-divestitures point may arise if small levels of divestitures raise the merged firm's marginal cost more than they reduce the marginal costs of the outsiders, leading to a reduction in market-level output. In such a case, for consumer surplus to be increased slightly above the no-divestitures level therefore requires a discrete level of divestitures, inducing a discrete drop in the merged firm's profit.

Solution It is straightforward to use the above framework to solve for the *minimum divestitures* $\boldsymbol{\Delta}_{\min}$ that are required to meet the authority's market-level consumer surplus constraint. Three different cases may arise. First, it may be that

$$\overline{v}_{ND}\ge xv(Q^*). \tag{7}$$

In this case, no divestitures are required to meet the constraint, and thus $\mathbf{\Delta}_{\min} = 0$. Second, it may be that

$$xv(Q^*) > \overline{v}_{\max}(> \overline{v}_{ND}). \tag{8}$$

In this case, there exist no feasible divestitures such that the market-level constraint is met, and hence the merger is blocked. Finally, it may be that

$$\overline{v}_{\max} \geq xv(Q^*) > \overline{v}_{ND}. \tag{9}$$

In this case the market-level constraint would be violated absent divestitures, but there do exist feasible divestitures such that it can be met. The firm's preferred outcome is then a divestiture which induces the point on the efficient divestitures curve that just meets the constraint. Equivalently, the minimum divestitures are such that $v(\overline{Q}^*(\mathbf{\Delta}_{\min})) = xv(Q^*)$. This follows because the efficient divestitures curve $d_M(v)$ is decreasing and has a connected support $[\overline{v}_{ND}, \overline{v}_{\max}]$.[10] The outcome associated with these minimum divestitures is depicted as a square in both Figures 2 and 4. Notice that in Figure 2 the merging parties are better off than before the merger even after offering divestitures, whereas in Figure 4 they are worse off. Nevertheless, recall that what matters is whether or not the merging parties gain from the merger (plus divestitures) after aggregating across all of the markets.

In the remainder of this section we assume that $\overline{v}_{\max} \geq xv(Q^*)$ holds in each market, such that feasible divestitures exist to meet the constraints. In what follows it will be useful to define

$$\overline{v}_{\min} \equiv \max\{\overline{v}_{ND}, xv(Q^*)\}, \tag{10}$$

which is the consumer surplus that arises in a given market at the minimum divestitures.

4.2 Cross-Market Comparison

We now compare the level of minimum divestitures across markets. To this end, consider two different markets j and k. To ensure that these markets are comparable, we make the following assumptions. Firstly, the merging parties are active in both markets, with the same pre-merger costs i.e. $c_i^j = c_i^k$ for each $i \in \mathcal{M}$. Secondly, the divestible assets are the same in both markets, as is the effect of divestitures on the merged firm's cost function i.e. $K^j = K^k = K$, and $\overline{c}_M^j(D) = \overline{c}_M^k(D)$ for any feasible vector D of divested assets. Thirdly, divestitures are equally effective in both markets. Precisely, when the vector D of divested assets is allocated in the most efficient way possible, the reduction in outsiders' summed costs is the same in both markets, i.e. $\min_{\mathbf{\Delta}^l \text{s.t.} \sum_{i \in \mathcal{O}^l} \Delta_i^l = D} \sum_{i \in \mathcal{O}^l} \left[\overline{c}_i^l(\Delta_i^l) - c_i^l\right]$ is the same for $l = j, k$. Moreover, the antitrust authority applies the same consumer surplus 'standard' in both markets, i.e. $x^j = x^k = x$. To simplify the analysis, we also assume that all firms present

[10] We are implicitly assuming here that $d_M\left(xv(Q^*)\right) > 0$, otherwise the merging parties would also be willing to offer divestitures (should they exist) that strictly meet the authority's constraint.

in market $l \in \{j, k\}$ produce a strictly positive output pre-merger as well as post-merger, i.e., $c_i^l < P^l(Q^{l*})$ as well as $\overline{c}_i^l(0) < P^l(\overline{Q}^{l*})$.[11]

We start by explaining what it means for minimum divestitures to be "lower" in one market than another:

Definition 1. *We say that required divestitures are lower in market j than in market k if the following are true:*

1. *If divestitures are required in market j, they are also required in market k.*

2. *If a vector D of divestitures weakly meets the market k constraint, those divestitures can be allocated in market j in such a way that market j's constraint is strictly met.*

Notice that, except in the special case where the vector of divestible assets K is one-dimensional, Definition 1 does not imply that the *optimal* minimum divestitures is such that weakly less of each asset (and strictly less of at least one) is divested in one market compared to another.

We now begin by analyzing the case where $x = 1$ i.e. the authority requires that in each of the two markets, the merger does not strictly reduce consumer surplus. Suppose that market j is more competitive in the sense that its pre-merger price is lower i.e. $P^j(Q^{j*}) < P^k(Q^{k*})$. As a first step, consider which of markets j and k is more likely to require divestitures. Using the equilibrium conditions (3) and (4) from earlier, it is straightforward to show that the constraint $v^l(\overline{Q}^{l*}) \geq v^l(Q^{l*})$ is satisfied in market $l \in \{j, k\}$ *absent divestitures* if and only if

$$\overline{c}_M^l(0) - \sum_{i \in \mathcal{M}} c_i^l + (n^l - \overline{n}^l) P^l(Q^{l*}) \leq 0, \tag{11}$$

which is just the formula for a consumer surplus increasing merger from the classic analysis of Farrell and Shapiro (1990). Notice that this inequality is strictly easier to satisfy in market j than in market k, and so the former is strictly less likely to require divestitures.

Continuing with the case $x = 1$, suppose that in fact divestitures are required in both markets. Using the equilibrium conditions (3) and (4) again, a divestiture $\mathbf{\Delta}^l$ in market $l \in \{j, k\}$ meets the constraint if and only if

$$\underbrace{\overline{c}_M^l\left(-\sum_{i \in \mathcal{O}^l} \Delta_i^l\right) + \sum_{i \in \mathcal{O}^l} \overline{c}_i^l\left(\Delta_i^l\right) - \sum_{i \in \mathcal{N}^l} c_i^l}_{\text{Equal in both markets under best asset allocation}} + (n^l - \overline{n}^l) P^l\left(Q^{l*}\right) \leq 0. \tag{12}$$

Suppose that divested assets are allocated in the best way amongst the outsider firms in each market. Then the above inequality is again strictly easier to satisfy in the more competitive

[11] Outsider $i \in \mathcal{O}$ in market l produces a positive output in post-merger equilibrium if $\overline{c}_M^l(0)$ is not too low and x is not too large.

market j. Consequently, if a vector of divestitures D weakly meets the constraint in market k, it strictly meets the constraint in market j. In summary, we find that:

Proposition 1. *Suppose that $x = 1$, and that market j is more competitive pre-merger than market k, in that $P^j(Q^{j*}) < P^k(Q^{k*})$. Then required divestitures are lower in market j in the sense of Definition 1.*

Proposition 1 provides a very general result, in that it allows both demand functions and the set of merger outsiders to vary arbitrarily across the two markets. To get some intuition for the result, note that a merger benefits consumers whenever the "efficiency effect" outweighs the "market power effect". The market power effect is strictly smaller in market j because the merger partners' pre-merger revenue is smaller, implying that the post-merger internalization of competitive externalities has a smaller effect.

We now analyze the case where the authority uses a more general constraint x in the two markets. In order to extend the result from Proposition 1, we specialize to the case where $P^j(Q) = P^k(Q) = P(Q)$, and market j has weakly more firms i.e. $n^j \geq n^k$. We obtain the following result:

Proposition 2. *Suppose market j is more competitive pre-merger than market k, in that $P(Q^{j*}) < P(Q^{k*})$, where $P(Q)$ is the common demand curve. Suppose also that $\sigma'(Q) \leq 0$ for all Q such that $P(Q) > 0$, and that $n^j \geq n^k$. Then there exists an $\hat{x} > 1$ such that:*

1. *If $x < \hat{x}$, then required divestitures are lower in market j.*
2. *If $x > \hat{x}$, then required divestitures are lower in market k.*

One might have the intuition that, in a more competitive market, firms have less market power so that a merger is less detrimental to consumers. As a result, one may naturally expect fewer divestitures (if any) to be required in a more competitive market. Propositions 1 and 2 show that this intuition is correct when the authority's x is not too large – but that the intuition fails for sufficiently high x. The reason is simply that more competitive markets start with higher consumer surplus pre-merger, and thus post-merger consumer surplus must also be higher – in which case more divestitures may be required. We also note that the regularity condition $\sigma'(Q) \leq 0$ on demand is satisfied by all demands with constant curvature such as linear demand, as well as other demands derived from common distributions such as Normal, Logistic, Type I Extreme Value, and Laplace.

Remark 1. *Propositions 1 and 2 extend to the case where assets can also be divested to a new firm which was previously not present in the two markets.*

To simplify the exposition, we assumed in this subsection that assets can only be divested to firms that were already active in the two markets pre-merger. However this is not necessary – Propositions 1 and 2 both extend to the case where divestitures can be used to create a brand new player in the two markets.

5 Multimarket Analysis

We now consider the multimarket bargaining problem. The negotiated divestitures must ensure that both the antitrust authority and the merging parties find the merger desirable. The antitrust authority only approves the merger (and associated divestitures) if all the market-level constraints are met, i.e. if $v(\overline{Q}^{j*}) \geq x^j v(Q^{j*})$ in each market $j \in [0,1]$. The merging parties only proceed with the merger if, after implementing the negotiated divestitures, their profit would be higher than before the merger, i.e. if $\overline{\Pi}_M^* \geq \sum_{i \in \mathcal{M}} \Pi_i^*$. To make the problem non-trivial we make two assumptions. First, we assume that $\overline{v}_{\max}^j \geq x^j v(Q^{j*})$ in each market $j \in [0,1]$, such that there do exist feasible divestitures that meet the authority's market-level constraints. Second, we also assume that $\int_{[0,1]} d_M^j(\overline{v}_{\min}^j)dj \geq \sum_{i \in \mathcal{M}} \Pi_i^*$, such that the merging parties would prefer to merge if they only need to implement the minimum divestitures.[12]

Using our earlier analysis, the solution to the bargaining problem must induce some aggregate consumer surplus V which satisfies

$$\int_{[0,1]} \overline{v}_{\min}^j dj \leq V \leq \int_{[0,1]} \overline{v}_{\max}^j dj. \tag{13}$$

We assume that bargaining is efficient. This means that conditional on the merger (and divestitures) implementing an aggregate consumer surplus V, the post-merger outcome will be on the efficient divestitures curve in each market. Hence the merged firm's profit aggregated over all markets is

$$\overline{\Pi}_M^*(V) \equiv \max_{(\tilde{v}^j \in [\overline{v}_{\min}^j, \overline{v}_{\max}^j])^{j \in [0,1]}} \int_{[0,1]} d_M^j(\tilde{v}^j)dj \quad \text{s.t.} \quad \int_{[0,1]} \tilde{v}^j dj = V. \tag{14}$$

The remainder of this section now proceeds as follows. In Section 5.1 we fix a given aggregate consumer surplus V, and characterize the multi-market divestitures which solve (14). In Section 5.2 we analyze properties of $\overline{\Pi}_M^*(V)$, as well as the V which satisfy the merging firms' participation constraint, i.e. which satisfy $\overline{\Pi}_M^*(V) \geq \sum_{i \in \mathcal{M}} \Pi_i^*$. Finally in Section 5.3 we study how the optimal divestitures vary across markets according to their pre-merger competitiveness.

5.1 Characterization

We start by fixing the aggregate consumer surplus V and characterize the optimal negotiated divestitures in each market. It is convenient to rewrite the constrained maximization problem

[12] Recall from Lemma 3 that the (efficient) divestitures curve in each market is weakly decreasing in v. Hence if the merger is not profitable at the minimum divestitures, it is also not profitable at any other feasible divestitures which satisfy the market-level constraints.

in equation (14) as

$$\mathcal{L} = \max_{(\tilde{v}^j \in [\overline{v}^j_{\min}, \overline{v}^j_{\max}])^{j\in[0,1]}} \int_{[0,1]} \left[d^j_M(\tilde{v}^j) + \lambda(\tilde{v}^j - V) \right] dj, \tag{15}$$

where $\lambda > 0$ is the Lagrange multiplier on the aggregate consumer surplus constraint. Notice that the optimization problem boils down to choosing consumer surplus $\tilde{v}^j$ in each market $j \in [0,1]$. Market j's contribution to the Langrangian is $d^j_M(\tilde{v}^j) + \lambda(\tilde{v}^j - V)$. Its derivative with respect to $\tilde{v}^j$ is positive if and only if $\lambda > -d^j_M(\tilde{v}^j)$ i.e. the (absolute value of the) slope of the efficient divestitures curve is less than the Lagrange multiplier λ. Henceforth we refer to $-d^j_M(\tilde{v}^j)$ as the *remedies exchange rate*, because it indicates how many dollars the merging partners have to give up in market j in order to increase consumer surplus in that market by one dollar.

Licensing / redundant assets case It is again instructive to start with the special case of licensing / redundant assets. Recall from earlier that different points along the divestitures curve correspond to different post-merger output levels. Using the profit fitting-in function, the effect of a small change in output Q on the merged firm's profit in market j (fixing its cost) can be written as

$$\frac{d\pi^j(Q; \overline{c}^j_M)}{dQ} = QP^{j\prime}(Q) s^j_M(Q) \left[2 - s^j_M(Q)\sigma^j(Q) \right], \tag{16}$$

where $s^j_M(Q) \equiv r^j(Q; \overline{c}^j_M)/Q$ is the merged firm's market share. At the same time, the effect of a small change in output Q on consumer surplus is $dv^j(Q)/dQ = -QP^{j\prime}(Q)$. Combining these, the remedies exchange rate in market j is given by

$$-d^{j\prime}(v) = s^j_M(Q^j(v)) \left[2 - s^j_M(Q^j(v))\sigma^j(Q^j(v)) \right] > 0, \tag{17}$$

where $Q^j(v)$ is the output in market j such that consumer surplus in that market equals v. (The inequality follows from the assumption that $\sigma^j(Q) < 1$ and the fact that $s^j_M(Q^j(v)) \leq 1$ in equilibrium.) We now impose the following regularity condition on market demand:

Condition 1. *For all Q such that $v(Q) \in [\overline{v}_{\min}, \overline{v}_{\max}]$, market demand satisfies*

$$2[1 - \sigma(Q)][2 - \sigma(Q)] + Q\sigma'(Q) > 0.$$

Condition 1 holds provided that $\sigma'(Q)$ is not too negative, and therefore ensures that inverse demand does not become "too concave" as market-level output increases. The condition is trivially satisfied by any demand function with constant curvature, such as linear demand. It is also satisfied by demands that are derived from many common distributions. For example, if demand is proportional to $1 - F(p)$, Condition 1 is satisfied if F is the c.d.f. of the standard Normal, Logistic, Type I Extreme Value, or Laplace distribution.

Proposition 3. *Suppose that in each market, either Condition 1 holds, or $s_M(Q)$ is sufficiently small for all Q such that $v(Q) \in [\overline{v}_{\min}, \overline{v}_{\max}]$. Then the solution to the multimarket bargaining problem is "bang-bang": in each market, consumer surplus is either $\overline{v}_{\min}$ or $\overline{v}_{\max}$.*

Proposition 3 shows that the optimal solution to the bargaining problem is bang-bang. Specifically, it is optimal to do only the minimal divestitures in a subset of the markets, and then to maximize consumer surplus in the remaining markets by doing as many divestitures there as possible. Intuitively, when the conditions in the hypotheses of Proposition 3 hold, the efficient divestitures curve is convex in v, i.e. $d^{j\prime\prime}(v) > 0$. This implies that the remedies exchange rate in any given market improves as more and more (consumer surplus increasing) divestitures are implemented there. Consequently, it cannot be optimal to have intermediate divestitures in a positive measure of markets – because a Pareto improvement can be obtained by doing fewer divestitures in some and more in others.

We now explain why the remedies exchange rate improves as v increases. First, note that consumer surplus is convex in output. As such, as v increases, an additional dollar increase in v can be achieved through successively smaller increases in output Q. Secondly, in the special case where market demand is linear, a unit increase in Q reduces market price by the same amount. However when Q is higher, the merged firm's market share is lower, and so it is hurt less by any given reduction in the market price. This explains why the exchange rate improves in v when demand is linear. Thirdly, when demand is non-linear a given increase in output Q does not reduce market price by the same amount. In particular, it is possible that as Q increases, price falls by more and more, such that the exchange rate worsens. This does not happen when either Condition 1 holds, or the merged firm's market share is sufficiently small. Condition 1 ensures that as Q increases demand does not become too concave, and so price does not fall too quickly. A low $s_M(Q)$ ensures that even if price does fall quickly, its effect on the firm's profit is dominated.

General case We now turn to the general case, where divesting assets may increase the merged firm's marginal cost. For simplicity, we assume that the set of active firms in each market does not vary with divestitures. Under that assumption, an efficient set of divestitures simply minimizes the sum of the active firms' marginal costs, for a given level of the merged firm's marginal cost. In other words, if the merged firm's cost is c, summed costs in market j are $\overline{C}^j(c)$ where

$$\overline{C}^j(c) \equiv \min_{\boldsymbol{\Delta}^j} \overline{c}_M^j \left(- \sum_{i \in \mathcal{O}^j} \Delta_i^j \right) + \sum_{i \in \mathcal{O}^j} \overline{c}_i^j(\Delta_i^j) \text{ s.t. } \overline{c}_M^j \left(- \sum_{i \in \mathcal{O}^j} \Delta_i^j \right) = c. \tag{18}$$

In order to derive the analog of Proposition 3 we impose some structure on this function:

Definition 2. *Assets are said to be complementary if $\overline{C}''(c) < 0$ for all c which satisfy $\overline{c}_M \left(- \sum_{i \in \mathcal{O}} (\Delta_{\min})_i \right) \leq c \leq \overline{c}_M(-K)$.*

A simple example of when assets are complementary in the sense of Definition 2 is as follows. Suppose that assets are one-dimensional, and that each firm has increasing returns from assets. It is straightforward to argue that it is better to give all divested assets to one outsider, rather than to multiple outsiders. Moreover since each firm has increasing returns, as the merged firm divests more assets its cost increases by less and less, whilst the cost of the outsider that receives the assets decreases by more and more. Equivalently, $\overline{c}_i^j(\Delta_i^j)$ is a concave function of $\overline{c}_M^j(-\Delta_i^j)$ for each outsider firm $i \in \mathcal{O}^j$. Since $\overline{C}^j(c)$ is the minimum over $(\overline{n}^j - 1)$ concave functions, it itself is concave.

We now derive the slope of the efficient divestitures curve. An important difference with the licensing / redundant assets case is that, in general, the merged firm's cost increases as it moves down that curve. Summing up firms' post-merger first order conditions yields

$$\overline{C}^j(\overline{c}_M^j) = \overline{n}^j P^j(Q) + QP^{j\prime}(Q), \tag{19}$$

and applying the implicit function theorem then gives

$$\frac{\partial \overline{c}_M^j(Q)}{\partial Q} = P^{j\prime}(Q)\frac{\overline{n}^j + 1 - \sigma^j(Q)}{\overline{C}^{j\prime}(\overline{c}_M^j(Q))}. \tag{20}$$

Using this equation, and following similar steps as in the licensing / redundant assets case, we can write the remedies exchange rate in market j as

$$-d^{j\prime}(v) = s_M^j(Q^j(v))\left[2 - s_M^j(Q^j(v))\sigma^j(Q^j(v))\right] - 2s_M^j(Q^j(v))\frac{\overline{n}^j + 1 - \sigma^j(Q^j(v))}{\overline{C}^{j\prime}(\overline{c}_M^j(Q(v))}, \tag{21}$$

where $Q^j(v)$ is again the output in market j such that consumer surplus in that market equals v. Notice that even if assets are complementary in the sense of Definition 2, the summed industry costs $\overline{C}^j(\overline{c}_M^j)$ may increase in $\overline{c}_M^j$, but if they start decreasing they continue to be decreasing for all higher $\overline{c}_M^j$. It is straightforward to prove from (21) that when $\overline{C}^{j\prime}(\overline{c}_M^j) > 0$ the remedies exchange rate is negative[13], and that when $\overline{C}^{j\prime}(\overline{c}_M^j) < 0$ it is positive.[14] Hence the supremum of the conditional divestitures curves can be backward-bending, causing a jump down in the efficient divestitures curves at the no-divestitures outcome, as depicted earlier in Figure 4. We now impose an additional regularity condition on market demand:

Condition 2. *For all Q such that $v(Q) \in [\overline{v}_{\min}, \overline{v}_{\max}]$, market demand satisfies*

$$[\overline{n} + 1 - \sigma(Q)] + Q\sigma'(Q) > 0.$$

This condition again ensures that demand does not become too concave as output increases, and is satisfied by many common demand functions.[15] We then obtain the following result:

[13]Note that by definition $\overline{C}^{j\prime}(\overline{c}_M^j) \leq 1$, which places a (negative) upper bound on the exchange rate.

[14]In the limit case $\overline{C}^{j\prime}(\overline{c}_M^j) \to -\infty$ this simplifies to the expression in the licensing / redundant assets case.

[15]It is again trivially satisfied for any demand with constant curvature, as well as demands that are derived

Proposition 4. *Assume assets are complementary. Suppose also that in each market, either Conditions 1 and 2 jointly hold, or $s_M(Q)$ is sufficiently small for all Q such that $v(Q) \in [\overline{v}_{\min}, \overline{v}_{\max}]$. Then the solution to the multimarket bargaining problem is again "bang-bang": in each market, consumer surplus is either $\overline{v}_{\min}$ or $\overline{v}_{\max}$.*

The hypotheses of Proposition 4 again ensure that the remedies exchange rate improves in the market-level consumer surplus v, leading to a bang-bang solution. Intuitively, as remarked earlier, asset complementarity ensures that if the efficient divestitures curve jumps down (implying an infinitely bad exchange rate) this occurs at the beginning i.e. at $v = \overline{v}_{ND}$. At higher levels of market-level consumer surplus $v \geq \overline{v}_{\min} (\geq \overline{v}_{ND})$, asset complementarity helps ensure that the merged firm's price-cost margin does not fall too quickly in aggregate output Q, implying that the remedies exchange rate improves.

Using Propositions 3 and 4 we can now fully characterize the solution to the multimarket bargaining problem. Given a particular value $\lambda > 0$ for the Lagrange multiplier, in market j it is optimal to choose whichever of $\overline{v}^j_{\min}$ and $\overline{v}^j_{\max}$ contributes more to the Lagrange multiplier. Consequently it is optimal to do the *maximal divestitures* possible if and only if

$$d^j_M(\overline{v}^j_{\max}) + \lambda \overline{v}^j_{\max} > d^j_M(\overline{v}^j_{\min}) + \lambda \overline{v}^j_{\min}, \tag{22}$$

which can be rearranged to give

$$a^j \equiv \frac{d^j_M(\overline{v}^j_{\min}) - d^j_M(\overline{v}^j_{\max})}{\overline{v}^j_{\max} - \overline{v}^j_{\min}} < \lambda. \tag{23}$$

Hence it is optimal to do as many divestitures as is feasible in market j if and only if the *average remedies exchange rate* a^j in that market is less than the Lagrange multiplier. To pin down the value of the Lagrange multiplier at the optimal solution, it then suffices to solve the following equation

$$\int_{[0,1]} \{\overline{v}^j_{\min} \mathbf{1}_{\{a^j > \lambda\}} + \overline{v}^j_{\max} \mathbf{1}_{\{a^j < \lambda\}}\} dj = V, \tag{24}$$

where V is the 'target' level of aggregate consumer surplus.[16] Finally, it is easy to see that if V increases from V' to $V'' > V'$, the Lagrange multiplier also increases. Hence the set of markets with maximal divestitures at $V = V''$ is a strict superset of those at $V = V'$.

from the standard Normal, Logistic, Type I Extreme Value, and Laplace.

[16] The above discussion assumes for simplicity that the set of markets with the same a^j has zero measure. If a strictly positive measure of markets has the same a^j, it may be necessary to mix and do maximal divestitures in some and minimal divestitures in others, in which case equation (24) is slightly different.

5.2 Bargaining Frontier

We now examine the set of outcomes over which the antitrust authority and merging parties bargain. We begin with the following straightforward result:

Proposition 5. *Suppose that the conditions for Propositions 3 and 4 are satisfied. Then $\overline{\Pi}_M^*(V)$ is strictly decreasing in V, and also weakly (generically strictly) concave in V.*[17]

We have already pointed out that as aggregate consumer surplus increases, maximal divestitures are implemented in more markets. Hence an increase in V leads to a reduction in the merged firm's profit $\overline{\Pi}_M^*(V)$. Moreover, since these additional divestitures are implemented in markets with a less favorable remedies exchange rate, a given increase in V reduces $\overline{\Pi}_M^*(V)$ by more when V is higher. Proposition 5 implies that there exists a decreasing and concave *bargaining frontier* in $(V, \overline{\Pi}_M^*(V))$-space, which we depict in Figure 5 using the solid curve. The shaded area corresponds to other possible (dominated) outcomes. These arise when each individual market is on its efficient divestitures curve, and yet those divestitures are sub-optimal because they do not satisfy the solution that we characterized earlier in equation (23). The dotted line in the figure represents the merging firms' pre-merger profit, and any negotiated outcome must lie above it.

We now briefly return to our discussion in the introduction about different approaches that an antitrust authority might take. First, the authority might follow a strict 'market-by-market' approach, and only require that the merger (plus divestitures) satisfies all the market-level constraints. In this case the negotiated remedies are simply the minimum divestitures $(\boldsymbol{\Delta}^j)^{j\in[0,1]}$ that we characterized in Section 4. This outcome is depicted as the circle in Figure 5. Second, the authority might deviate from a strict 'market-by-market' rule. In particular, it might still require the market-level constraints to hold, but might use its bargaining power to increase aggregate consumer surplus above $\int_{[0,1]} \overline{v}^j_{\min} dj$. In terms of Figure 5, the negotiated solution then lies somewhere on the solid curve between the circle and the square. As the authority's bargaining power increases, V goes up and so maximal divestitures are implemented in more markets, reducing the merged firm's profit $\overline{\Pi}_M^*$. Finally, another possibility is that the antitrust authority *only* cares about aggregate consumer surplus, and is both willing and able to fully balance pro- and anti-competitive effects of the merger across markets. This would be the same as the second case but with $x^j = 0$ for all $j \in [0,1]$ i.e. the merger is allowed to (arbitrarily) reduce consumer surplus in some markets as long as it bring sufficient benefits in others.

5.3 Cross-Market Comparison

We now examine how a market's (pre-merger) competitiveness affects whether it should optimally have the minimum or the maximum divestitures. Closely following Section 4.2, consider two comparable markets j and k. Specifically, assume that i) the merging parties

[17] Genericity here refers to a zero measure of markets having the same average remedies exchange rate a^j.

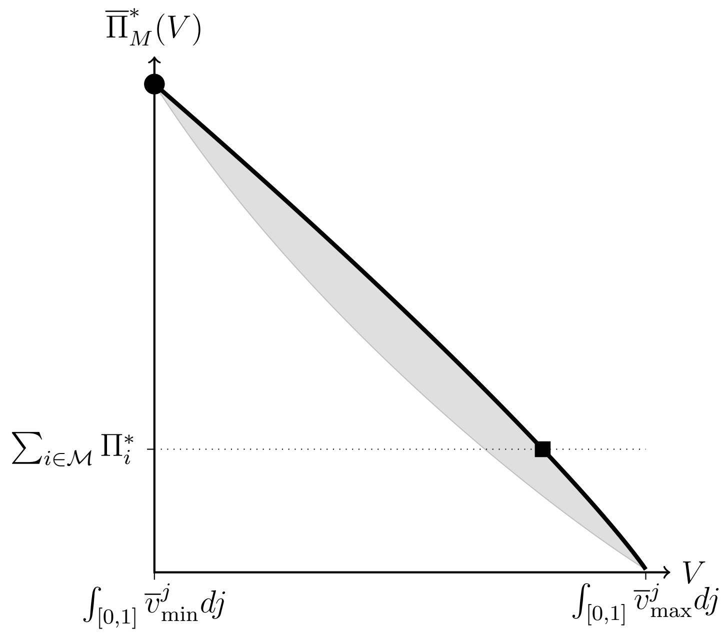

Figure 5: Possible solutions to the bargaining problem.

are active in both markets with the same pre-merger costs, ii) merged-induced synergies are identical in the two markets, as are the divestible assets, iii) divestitures have the same effect in both markets on the merged firm's cost and the summed costs of outsiders, iv) the markets have the same demand function $P(Q)$, and the authority applies the same x in each of them, and v) outsiders to the merger are active in each market pre- and post-merger. It is again instructive to begin by analyzing the licensing / redundant assets case.

Licensing / redundant assets case Suppose that market j is more competitive than market k in the following sense: there is an injective function $h(i)$ such that for every firm i in market k, there is another firm $h(i)$ in market j such that $c_i^k \geq c^j_{h(i)}$, with at least one strict inequality. This implies that market j has weakly more firms and a strictly lower pre-merger price than market k. We obtain the following result:

Proposition 6. *Suppose Condition 1 holds, and market j is more competitive than market k. Then the average remedies exchange rate is smaller in market j, i.e. $a^j < a^k$. Hence:*

- *If the bargaining solution entails maximal divestitures in market k, it also entails maximum divestitures in the more competitive market j.*

- *If the bargaining solution entails minimal divestitures in market j, it also entails minimal divestitures in the less competitive market k.*

Recall from Propositions 1 and 2 that, assuming x is not too large, more competitive markets need fewer divestitures to meet the market level constraint. Interestingly however, Proposition 6 implies that in the solution to the multimarket bargaining problem, more competitive markets should typically have *more* divestitures. To get some intuition for this striking result, recall from earlier that the remedies exchange rate in market l is

$$-d^{l\prime}(v) = s_M^l(Q^l(v)) \left[2 - s_M^l(Q^l(v))\sigma^l(Q^l(v))\right]. \tag{25}$$

Notice that, other things being equal, the remedies exchange rate is increasing in the merged firm's market share. Hence in competitive markets – where that market share is smaller – the merged firm is hurt less by asset divestitures (and the resulting decrease in the market price). Divesting assets in such markets is thus a 'cheap' way for the merged firm to deliver increased aggregate consumer surplus.

General case We now consider to what extent Proposition 6 extends to the general case, where divesting assets may increase the merged firm's marginal cost. To simplify the analysis, we now say that market j is more competitive than market k if the sum of the firms' pre-merger marginal costs is strictly lower in the former.

We begin our analysis in the case where the market-level constraint is met in both markets even absent divestitures. Here, we obtain the following result:

Proposition 7. *Suppose Conditions 1 and 2 hold, and that market j is more competitive than market k. If each market's constraint is met absent divestitures, then $a^j < a^k$.*

Proposition 7 shows that in the special case where the merger satisfies the constraint in both markets even without divestitures, more competitive markets again have a lower remedies exchange rate. Consequently, it is again the case that if, for example, it is optimal to do maximal divestitures in the less competitive market, it is also optimal to do them in the more competitive one as well.

It turns out, however, that the result in Proposition 7 does *not* necessarily generalize to the case where one or both markets would fail the constraint in the absence of divestitures. To get some intuition for why, consider an extreme example where, absent divestitures, the more competitive market j would just meet the constraint while the less competitive market would just fail it. Further suppose that the efficient divestitures curve exhibits a downward jump at the no-divestitures outcome, as for example depicted earlier in Figure 4. It is straightforward to see that if the maximal outputs $\overline{v}^j_{\max}$ and $\overline{v}^k_{\max}$ are similar, then the remedies exchange rate is more favorable in the less competitive market, i.e. $a^k < a^j$. This is because to meet the constraint in market k the merging parties have already needed to undertake the initial very costly divestitures, implying that it is relatively cheap to then do further divestitures in order to move from $\overline{v}^k_{\min}$ to $\overline{v}^k_{\max}$. On the other hand, the merged firm avoids the initial and very costly divestitures in market j if it just wishes to meet the constraint there, implying that they need to be done in order to move from $\overline{v}^j_{\min}$ to $\overline{v}^j_{\max}$ – implying that such a move is very costly. Although this is a deliberately extreme example, it is straightforward to construct other examples where $a^k < a^j$ even when the efficient divestitures curve does not exhibit a jump, and when both markets fail the constraint absent divestitures. In all cases, the key idea is that if $x < \hat{x}$, the less competitive market requires higher divestitures just to meet the constraint. Since the (continuously differentiable part of the) efficient divestitures curve is convex, this means that the merged firm has already undertaken the 'more costly' divestitures in the less competitive market in order to meet the constraint. This again implies that moving from $\overline{v}^k_{\min}$ to $\overline{v}^k_{\max}$ can be relatively cheaper than the equivalent move in the more competitive market j.

6 Conclusion

This paper develops a framework to think about remedies in the context of mergers which affect several different product or geographic markets. Using this framework, we showed that under a so-called 'market-by-market' rule, markets that are more competitive prior to the merger usually require fewer (if any) divestitures. We also showed that when the antitrust authority deviates from such a rule, and potentially allows remedies to be 'balanced' across markets, the solution is typically bang-bang. In particular, it is optimal to implement largescale divestitures in a small subset of affected markets, rather than do a small number of divestitures in many markets.

Our paper opens up several avenues for future research. Firstly, for simplicity, we have assumed throughout the paper that divested assets generate no revenue for the merging parties. However, our preliminary analysis suggests that this is not crucial for the main results. Secondly, we have assumed that markets are independent both in terms of demand and assets. It would be natural to see how far the results extend when, for example, some assets are used in several markets. Thirdly, we assumed that firms compete à la Cournot, and it would be interesting to also consider a model of price competition with differentiated products. Finally, we also assumed that both the antitrust authorities and the merging parties have full information about the effect of any divestitures. It would be interesting to endow the merging parties with some private information about the quality of their assets, and hence the efficacy of any divestiture.

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A Appendix

Proof of Lemma 1. We begin with the case $\overline{c}_M(0) < P\left(\overline{Q}^*\right)$. Consider a small increase in $\overline{c}_M(0)$. First, notice that aggregate output must strictly decrease. If on the contrary aggregate output were to weakly increase, the fitting-in function would strictly decrease for firm M and weakly decrease for each firm $i \in \mathcal{O}$, yielding a contradiction. Second, since aggregate output strictly decreases, so does consumer surplus. Third, consider M's profit. Suppose M's cost increases to $\overline{c}'_M(0) > \overline{c}_M(0)$, and let $\overline{Q}^{*\prime} < \overline{Q}^*$ be aggregate outputs associated with respectively $\overline{c}'_M(0)$ and $\overline{c}_M(0)$. Notice that

$$\begin{aligned}
& \left[P\left(r\left(\overline{Q}^*;\overline{c}_M(0)\right)+\sum_{i\in\mathcal{O}} r\left(\overline{Q}^*;\overline{c}_i(0)\right)\right)-\overline{c}_M(0)\right] r\left(\overline{Q}^*;\overline{c}_M(0)\right) \\
> & \left[P\left(r\left(\overline{Q}'^*;\overline{c}'_M(0)\right)+\sum_{i\in\mathcal{O}} r\left(\overline{Q}^*;\overline{c}_i(0)\right)\right)-\overline{c}_M(0)\right] r\left(\overline{Q}'^*;\overline{c}'_M(0)\right) \\
\geq & \left[P\left(r\left(\overline{Q}'^*;\overline{c}'_M(0)\right)+\sum_{i\in\mathcal{O}} r\left(\overline{Q}'^*;\overline{c}_i(0)\right)\right)-\overline{c}'_M(0)\right] r\left(\overline{Q}'^*;\overline{c}'_M(0)\right) .
\end{aligned}$$

The first inequality uses $r\left(\overline{Q}'^*;\overline{c}'_M(0)\right) < r\left(\overline{Q}^*;\overline{c}_M(0)\right)$ and the strict quasiconcavity of M's profit function.[18] The second inequality uses $r\left(\overline{Q}'^*;\overline{c}'_M(0)\right) \geq 0$, $\overline{c}'_M(0) > \overline{c}_M(0)$, and also the fact that since aggregate output strictly decreases it must be that each $i \in \mathcal{O}$ weakly increases its output. Hence M's profit strictly decreases.

Now consider the case $\overline{c}_M(0) \geq P\left(\overline{Q}^*\right)$. This inequality continues to hold for any increase in $\overline{c}_M(0)$. Therefore no firm's fitting-in function changes, and so total output and consumer surplus are unchanged, while M continues to make zero profit. $\square$

Proof of Lemma 2. Note that for any feasible divestitures $\mathbf{\Delta}$ there exists a unique $\overline{Q}^*$. First, since $\overline{c}_M(\cdot)$ is constant and $\overline{c}_i(\Delta_i)$ is weakly decreasing for each $i \in \mathcal{O}$, we must have $\overline{Q}^*(\mathbf{\Delta}) \geq \overline{Q}^*(0)$ for any feasible divestiture $\mathbf{\Delta}$, and thus $v \geq \overline{v}_{ND}$. Second, since the set of divestitures $\mathbf{\Delta}$ satisfying $\sum_{i\in\mathcal{O}} \Delta_i \leq K$ is compact, continuity of $v(Q)$ ensures that $\overline{v}_{\max}$ is well-defined. Third, continuity of $\overline{c}_i(\Delta_i)$ for each $i \in \mathcal{O}$ ensures that any $v \in [\overline{v}_{ND}, \overline{v}_{\max}]$ can be attained by some divestitures. Finally, note that $d_M(v) = \pi(Q(v);\overline{c}_M(\cdot))$, where $Q(v)$ is the inverse of $v(Q)$. The stated properties of $d_M(v)$ then follow immediately. $\square$

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[18]Note that since $\overline{Q}'^* < \overline{Q}^*$, we must have $r\left(\overline{Q}'^*;\overline{c}_i(0)\right) \geq r\left(\overline{Q}^*;\overline{c}_i(0)\right)$ for each $i \in \mathcal{O}$. These two facts necessarily imply $r\left(\overline{Q}'^*;\overline{c}'_M(0)\right) < r\left(\overline{Q}^*;\overline{c}_M(0)\right)$.

Proof of Lemma 3. First, note that by continuity of output fitting-in functions in cost, and continuity of cost functions in assets, any $v \in [\overline{v}_{ND}, \overline{v}_{\max}]$ can be achieved by some feasible vector of asset divestitures. The function $d_M(v)$ is then the maximum over those functions.

Second, we prove that $d_M(v)$ is strictly decreasing when $d_M(v) > 0$. Suppose not. Then there exist v' and v'', and associated divestitures $\mathbf{\Delta}'$ and $\mathbf{\Delta}''$, such that both $\overline{v}_{ND} \leq v' < v'' \leq \overline{v}_{\max}$ and $0 < d_M(v') \leq d_M(v'')$. However since conditional divestitures curves are strictly decreasing when $\tilde{d}_M(v; \cdot) > 0$, this implies that $d_M(v', \sum_{i \in \mathcal{O}} \Delta_i'') > d_M(v', \sum_{i \in \mathcal{O}} \Delta_i')$. This is a contradiction.

Finally, the proof for why $d_M(v)$ is non-increasing when $d_M(v) = 0$ is very similar and so is omitted. $\square$

Proof of Proposition 1. Given the discussion in the main text, it suffices to derive conditions (11) and (12). First, rewrite equations (3) and (4) for market $l \in \{j, k\}$ as

$$n^l P^l \left(Q^{l*}\right) + Q^{l*} P'^l \left(Q^{l*}\right) = \sum_{i \in \mathcal{N}^l} c_i^l, \tag{26}$$

$$\overline{n}^l P^l \left(\overline{Q}^{l*}\right) + \overline{Q}^{l*} P'^l \left(\overline{Q}^{l*}\right) = \overline{c}_M^l \left(-\sum_{i \in \mathcal{O}^l} \Delta_i^l\right) + \sum_{i \in \mathcal{O}^l} \overline{c}_i^l \left(\Delta_i^l\right). \tag{27}$$

The lefthand side of (27) is decreasing in $\overline{Q}^{l*}$, and so the constraint $\overline{Q}^{l*} \geq Q^{l*}$ implies

$$\overline{n}^l P^l \left(Q^{l*}\right) + Q^{l*} P'^l \left(Q^{l*}\right) \geq \overline{c}_M^l \left(-\sum_{i \in \mathcal{O}^l} \Delta_i^l\right) + \sum_{i \in \mathcal{O}^l} \overline{c}_i^l \left(\Delta_i^l\right). \tag{28}$$

Combining this with equation (26) and collecting terms yields condition (12). To derive condition (11), then impose $\Delta_i^l = 0$ for each $i \in \mathcal{O}$ and note that by definition $\overline{c}_i^l(0) = c_i^l$ for each $i \in \mathcal{O}$. $\square$

Proof of Proposition 2. Let $\overline{C}^l(x) \equiv \overline{c}_M^l(.) + \sum_{i \in \mathcal{O}^l} \overline{c}_i^l(.)$ be the summed post-merger costs which ensure that the constraint $v\left(\overline{Q}^{l*}\right) = xv\left(Q^{l*}\right)$ holds in market $l \in \{j, k\}$.

First, differentiate the constraint with respect to x, and use the equilibrium condition (4) for post-merger output $\overline{Q}^{l*}$ to obtain that

$$\begin{aligned} \overline{C}^{l\prime}(x) &= -\frac{v\left(Q^{l*}\right)}{\overline{Q}^{l*}} \left(\overline{n}^l + 1 - \sigma\left(\overline{Q}^{l*}\right)\right) \\ &= -\frac{1}{x} \frac{v\left(\overline{Q}^{l*}\right)}{\overline{Q}^{l*}} \left(\overline{n}^l + 1 - \sigma\left(\overline{Q}^{l*}\right)\right). \end{aligned}$$

Note that $v(Q)$ is strictly convex for all $Q > 0$, and recall that $\overline{n}^j \geq \overline{n}^k$ and $\sigma'(Q) \leq 0$ by assumption. Therefore since for a given x we have $\overline{Q}^{j*} > \overline{Q}^{k*}$, it immediately follows that $\overline{C}^{j\prime}(x) < \overline{C}^{k\prime}(x) < 0$.

Second, we know from Proposition 1 that $\overline{C}^j(1) - \sum_{i\in\mathcal{N}^j} c_i^j = \overline{C}^k(1) - \sum_{i\in\mathcal{N}^k} c_i^k$. Combined with the first step, this implies that there exists an $\hat{x}$ such that $\overline{C}^j(x) - \sum_{i\in\mathcal{N}^j} c_i^j > \overline{C}^k(x) - \sum_{i\in\mathcal{N}^k} c_i^k$ for $x < \hat{x}$, and such that the reverse holds for $x > \hat{x}$.

Third, consider $x < \hat{x}$. Suppose divestitures are required in market j, i.e.

$$\overline{c}_M^j(0) + \sum_{i\in\mathcal{O}^j} \overline{c}_i^j(0) > \overline{C}^j(x). \tag{29}$$

Combining this with the second step, and also using the assumptions that $\overline{c}_M^j(0) = \overline{c}_M^k(0)$ and also $c_i^j = c_i^k$ for each $i \in \mathcal{M}$, as well as the fact that $\overline{c}_i^l(0) = c_i^l$ for each $i \in \mathcal{O}^l$ and $l = j, k$, we obtain that

$$\overline{c}_M^k(0) + \sum_{i\in\mathcal{O}^k} \overline{c}_i^k(0) > \overline{C}^k(x), \tag{30}$$

which implies that divestitures are also required in market k. Now suppose that a vector of divestitures $D = \sum_{i\in\mathcal{O}^k} \Delta_i^k$ weakly meets the constraint for market k, i.e.

$$\overline{c}_M^k(-D) + \sum_{i\in\mathcal{O}^k} \overline{c}_i^k\left(\Delta_i^k\right) \leq \overline{C}^k(x). \tag{31}$$

Combine this with the second step, and also use the assumptions that $\overline{c}_M^j(0) = \overline{c}_M^k(0)$, that $c_i^j = c_i^k$ for each $i \in \mathcal{M}$, and also that divestitures are equally effective in the two markets. Then if divestitures $\boldsymbol{\Delta}^j$ satisfying $D = \sum_{i\in\mathcal{O}^j} \Delta_i^j$ are allocated in the best way in market j, it must be that

$$\overline{c}_M^j(-D) + \sum_{i\in\mathcal{O}^j} \overline{c}_i^j\left(\Delta_i^j\right) < \overline{C}^j(x), \tag{32}$$

which implies that the market j constraint is strictly met.

Finally, the proof for $x > \hat{x}$ is similar and so is omitted. □

Proof of Proposition 3. We first prove that $d^{j\prime\prime}(v) > 0$ when either Condition 1 holds, or $s_M^j(Q^j)$ is sufficiently small for all Q^j such that $v(Q^j) \in [\overline{v}_{\min}^j, \overline{v}_{\max}^j]$. Note that $d^{j\prime\prime}(v) > 0$ if and only if

$$-\frac{ds_M^j(Q^j)\left[2 - s_M^j(Q^j)\,\sigma^j(Q^j)\right]}{dQ^j}\frac{1}{v^{j\prime}(Q^j)} > 0, \tag{33}$$

where to simplify the exposition we omit the dependence of Q^j on v. Since $s_M^j(Q^j) = r(Q^j; \overline{c}_M^j)/Q^j$ we have that

$$\frac{ds_M^j(Q^j)}{dQ^j} = -\frac{1 + s_M^j(Q^j)\left[1 - \sigma^j(Q^j)\right]}{Q^j}. \tag{34}$$

Using this and the fact that $v^{j\prime}(Q^j) > 0$, (33) is equivalent to

$$2\left[\frac{1}{s_M^j(Q^j)} - \sigma^j\left(Q^j\right)\right]\left[\frac{1}{s_M^j(Q^j)} + 1 - \sigma^j\left(Q^j\right)\right] + Q^j\sigma^{j\prime}\left(Q^j\right) > 0. \tag{35}$$

As $s_M^j(Q^j) \to 0$ the lefthand side of (35) becomes unboundedly large and so the inequality holds. Alternatively, since the lefthand side of (35) is strictly decreasing in $s_M^j(Q^j)$, a sufficient condition for (35) to hold is that it is satisfied at $s_M^j(Q^j) = 1$. Substituting $s_M^j(Q^j) = 1$ into (35) gives Condition 1.

To complete the proof, note that by the previous step market j's contribution $d_M^j(\tilde{v}^j) + \lambda(\tilde{v}^j - V)$ to the Lagrange function is strictly convex in $\tilde{v}^j$. Hence the optimal $\tilde{v}^j$ is extremal, and thus equal to either $\overline{v}_{\min}^j$ or $\overline{v}_{\max}^j$. □

Proof of Proposition 4. We have already argued in the text that if the efficient divestitures curve has a discontinuity, it arises at $v = \overline{v}_{ND}$, and otherwise it is continuously differentiable. We first prove that under the hypotheses of the proposition, $d_M^{j\prime\prime}(v) > 0$ at all $v > \overline{v}_{\min}$. (Recall that by definition $\overline{v}_{\min} \geq \overline{v}_{ND}$.) Henceforth we drop the dependence of Q^j on v to simplify the exposition. Since $s_M^j(Q^j) = r\left(Q^j; \overline{c}_M^j(Q^j)\right)/Q^j$ we have that

$$\frac{ds_M^j(Q^j)}{dQ^j} = -\frac{1}{Q^j}\left[1 + s_M^j\left(Q^j\right)\left[1 - \sigma^j\left(Q^j\right)\right] - \frac{\overline{n}^j + 1 - \sigma^j(Q^j)}{\overline{C}^{j\prime}(\overline{c}_M)}\right]. \tag{36}$$

The first term of $-d_M^{j\prime}(v)$ (equation (21)) strictly decreases in Q^j if and only if

$$2\left[\frac{1}{s_M^j(Q^j)} - \sigma^j\left(Q^j\right)\right]\left[\frac{1}{s_M^j(Q^j)} + 1 - \sigma^j\left(Q^j\right) - \frac{\overline{n}^j + 1 - \sigma^j(Q^j)}{s_M^j(Q^j)\,\overline{C}^{j\prime}(\overline{c}_M)}\right] + Q^j\sigma^{j\prime}\left(Q^j\right) > 0. \tag{37}$$

Since $\overline{C}^{j\prime}(\overline{c}_M) < 0$ at any point on the efficient divestitures curve, (37) holds for $s_M^j(Q^j)$ sufficiently small or when Condition 1 holds. Meanwhile given that $\overline{C}^{j\prime}(\overline{c}_M) < 0$ and $\overline{C}^{j\prime\prime}(\overline{c}_M) < 0$, the second term in the expression for $-d_M^{j\prime}(v)$ is strictly decreasing in Q^j provided that $s_M^j(Q^j)\left[\overline{n}^j + 1 - \sigma^j(Q^j)\right]$ is strictly decreasing in Q^j. This is true if

$$\left[\overline{n}^j + 1 - \sigma^j\left(Q^j\right)\right]\left[\frac{1}{s_M^j(Q^j)} + 1 - \sigma^j\left(Q^j\right) - \frac{\overline{n}^j + 1 - \sigma^j(Q^j)}{s_M^j(Q^j)\,\overline{C}^{j\prime}(\overline{c}_M)}\right] + Q^j\sigma^{j\prime}\left(Q^j\right) > 0\ , \tag{38}$$

which is satisfied for $s_M^j(Q^j)$ sufficiently small or when Condition 2 holds.

To complete the proof, note that as in the proof of Proposition 3 the strict convexity of $d_M^j(\tilde{v})$ implies that market j's contribution to the Lagrange function is strictly convex in $\tilde{v}$, and hence either $\overline{v}_{\min}^j$ or $\overline{v}_{\max}^j$ is chosen. □

Proof of Proposition 5. Differentiating the Lagrangian with respect to V gives $\overline{\Pi}_M^{*\prime}(V) = -\lambda$. First, it is immediate that $\overline{\Pi}_M^{*\prime}(V) < 0$ because $\lambda > 0$. Second, to prove that $\overline{\Pi}_M^{*\prime\prime}(V) \leq 0$ (with generically a strict inequality) we need to show that λ is weakly (and generically strictly) increasing in V. To this end, consider $V_0, V_1 \in \left[\int_{[0,1]} \overline{v}_{\min}^j dj, \int_{[0,1]} \overline{v}_{\max}^j dj\right]$ which satisfy $V_0 < V_1$. Suppose to the contrary that the associated Lagrange multipliers satisfy $\lambda_0 > \lambda_1$: as shown in the text, the set of markets with maximal divestitures under λ_0 would be a superset of those under λ_1, contradicting $V_0 < V_1$. Hence we must have $\lambda_0 \leq \lambda_1$. To complete the proof, note that if $\lambda_0 = \lambda_1$ then generically the set of markets with maximal divestitures would be identical under λ_0 and λ_1, but this would again contradict $V_0 < V_1$. Hence generically $\lambda_0 < \lambda_1$. □

Proof of Proposition 6. First, note that given our assumptions,

$$\sum_{i \in \mathcal{N}^j} r\left(Q; c_i^j\right) > \sum_{i \in \mathcal{N}^k} r\left(Q; c_i^k\right), \tag{39}$$

which implies that pre-merger outputs satisfy $Q^{j*} > Q^{k*}$.

Second, it is straightforward to show that (39) and $\overline{c}_M^j(0) = \overline{c}_M^k(0)$ together imply

$$r\left(Q, \overline{c}_M^j(0)\right) + \sum_{i \in \mathcal{O}^j} r\left(Q; \overline{c}_i^j(0)\right) > r\left(Q, \overline{c}_M^k(0)\right) + \sum_{i \in \mathcal{O}^k} r\left(Q; \overline{c}_i^k(0)\right), \tag{40}$$

which in turn implies that absent divestitures $\overline{Q}^{j*}(0) > \overline{Q}^{k*}(0)$.

Third, we claim that $\overline{v}_{\min}^j > \overline{v}_{\min}^k$. If $\overline{v}_{\min}^k = xv\left(Q^{k*}\right)$ then the result follows from the first step, whereas if $\overline{v}_{\min}^k > xv\left(Q^{k*}\right)$ then the result follows from the second step.

Fourth, we claim that $\overline{v}_{\max}^j > \overline{v}_{\max}^k$. Since divestitures are equally effective in the two markets, it is straightforward to show that when the K assets are divested in the best way in each market, (40) implies that

$$r\left(Q, \overline{c}_M^j(-K)\right) + \sum_{i \in \mathcal{O}^j} r\left(Q; \overline{c}_i^j(.)\right) > r\left(Q, \overline{c}_M^k(-K)\right) + \sum_{i \in \mathcal{O}^k} r\left(Q; \overline{c}_i^k(.)\right). \tag{41}$$

This in turn implies that $\overline{Q}^{j*}(K) > \overline{Q}^{k*}(K)$ and thus $\overline{v}_{\max}^j > \overline{v}_{\max}^k$.

Fifth, note that since $\overline{c}_M^j(0) = \overline{c}_M^k(0)$ we can write that $d_M^j(v) = d_M^k(v) = d_M(v)$. Consequently for $l = j, k$ we have that

$$a^l = \frac{d_M\left(\overline{v}_{\min}^l\right) - d_M\left(\overline{v}_{\max}^l\right)}{\overline{v}_{\max}^l - \overline{v}_{\min}^l}. \tag{42}$$

It is straightforward to show that the strict convexity of $d_M(v)$ implies that a^l is strictly decreasing in both $\overline{v}^l_{\min}$ and $\overline{v}^l_{\max}$. Therefore using the third and fourth steps, we can write

$$a^j = \frac{d_M\left(\overline{v}^j_{\min}\right) - d_M\left(\overline{v}^j_{\max}\right)}{\overline{v}^j_{\max} - \overline{v}^j_{\min}} < \frac{d_M\left(\overline{v}^k_{\min}\right) - d_M\left(\overline{v}^j_{\max}\right)}{\overline{v}^j_{\max} - \overline{v}^k_{\min}} < \frac{d_M\left(\overline{v}^k_{\min}\right) - d_M\left(\overline{v}^k_{\max}\right)}{\overline{v}^k_{\max} - \overline{v}^k_{\min}} = a^k. \tag{43}$$

Finally, recall that as many divestitures as possible are done in market $l = j, k$ if and only if $a^l < \lambda$. The fifth step therefore implies that if maximal divestitures are done in market k (i.e. $a^k < \lambda$) they also also done in market j, and that if minimal divestitures are done in market j (i.e. $a^j > \lambda$) they are also done in market k. □

Proof of Proposition 7. Recall that $P^j(Q) = P^k(Q) = P(Q)$ and also $\overline{c}^j_M(-K) = \overline{c}^k_M(-K) = \overline{c}_M(-K)$. Hence we can write for market $l = j, k$ that

$$a^l = \frac{\tilde{d}_M\left(\overline{v}^l_{\min}; \overline{c}_M(0)\right) - \tilde{d}_M\left(\overline{v}^l_{\min}; \overline{c}_M(-K)\right)}{\overline{v}^l_{\max} - \overline{v}^l_{\min}} + \frac{\tilde{d}_M\left(\overline{v}^l_{\min}; \overline{c}_M(-K)\right) - \tilde{d}_M\left(\overline{v}^l_{\max}; \overline{c}_M(-K)\right)}{\overline{v}^l_{\max} - \overline{v}^l_{\min}}, \tag{44}$$

where, as defined earlier, $\tilde{d}_M(v; c)$ is a conditional divestitures curve.

First, using similiar steps as in the proof of Proposition 6, it is straightforward to show that $\overline{v}^j_{\min} > \overline{v}^k_{\min}$ and $\overline{v}^j_{\max} > \overline{v}^k_{\max}$.

Second, we claim that $\tilde{d}_M\left(\overline{v}^l_{\min}; \overline{c}_M(0)\right) - \tilde{d}_M\left(\overline{v}^l_{\min}; \overline{c}_M(-K)\right)$ is strictly lower for $l = j$ than for $l = k$. To prove this, note that

$$\begin{aligned}\tilde{d}_M\left(\overline{v}^l_{\min}; \overline{c}_M(0)\right) - \tilde{d}_M\left(\overline{v}^l_{\min}; \overline{c}_M(-K)\right) &= -\int_{\overline{c}_M(0)}^{\overline{c}_M(-K)} \frac{\partial \pi\left(Q\left(\overline{v}^l_{\min}\right); c\right)}{\partial c} dc \\ &= 2\int_{\overline{c}_M(0)}^{\overline{c}_M(-K)} r\left(Q\left(\overline{v}^l_{\min}\right); c\right) dc.\end{aligned}$$

Using $Q\left(\overline{v}^j_{\min}\right) > Q\left(\overline{v}^k_{\min}\right)$ (from the first step) and the fact that $r(Q; c)$ is strictly decreasing in Q whenever $r(Q; c) > 0$, the claim follows immediately.

Third, we also claim that $\overline{v}^l_{\max} - \overline{v}^l_{\min}$ is strictly higher for $l = j$ than for $l = k$. To prove this, note that there exists a $\tau > 0$ such that $\overline{C}^j(\overline{c}_M(0)) = \overline{C}^k(\overline{c}_M(0)) - \tau$, and that since divestitures are equally effective in both markets, $\overline{C}^j(\overline{c}_M(-K)) \leq \overline{C}^k(\overline{c}_M(-K)) - \tau$. Abusing notation and letting $Q(C)$ be the equilibrium output given summed market costs C, we can write that

$$\overline{v}^k_{\max} - \overline{v}^k_{\min} \leq v\left(Q\left(\overline{C}^j(\overline{c}_M(-K)) + \tau\right)\right) - v\left(Q\left(\overline{C}^j(\overline{c}_M(0)) + \tau\right)\right). \tag{45}$$

To prove the claim, it is then sufficient to show that the righthand side of (45) is decreasing in τ. Its derivative with respect to τ is

$$-\frac{Q\left(\overline{C}^{j}\left(\overline{c}_{M}\left(-K\right)\right)+\tau\right)}{\overline{n}+1-\sigma\left(Q\left(\overline{C}^{j}\left(\overline{c}_{M}\left(-K\right)\right)+\tau\right)\right)}+\frac{Q\left(\overline{C}^{j}\left(\overline{c}_{M}\left(0\right)\right)+\tau\right)}{\overline{n}+1-\sigma\left(Q\left(\overline{C}^{j}\left(\overline{c}_{M}\left(0\right)\right)+\tau\right)\right)},$$

which given Condition 2 is negative, because $\overline{v}^{j}_{\max} > \overline{v}^{j}_{\min}$ implies that $\overline{C}^{j}\left(\overline{c}_{M}\left(-K\right)\right) < \overline{C}^{j}\left(\overline{c}_{M}\left(0\right)\right)$.

The second and third steps combined imply that the first term in (44) is strictly lower in market j than in market k.

Finally, consider the second term in (44). Following the same steps as in the proof of Proposition 6, it is straightforward to show that this term is strictly decreasing in both $\overline{v}^{l}_{\min}$ and $\overline{v}^{l}_{\max}$, and thus using the first step, this term is strictly lower in market j than in market k. $\square$